MATRIX OPTICS OF CAMERA LENSES

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Given a set of optical components for a camera, we may wish to stack them in various orders to achieve effects outside the range of their individual capabilities. Assuming the physical aspects of a given setup are possible, we would like to know its net optical properties. There are various ways to go about this. Playing around with cardinal points and distances is a tedious exercise and prone to error, while direct ray tracing requires numerical simulation. The cleanest and safest approach is through matrix optics. Although we are limited by lack of knowledge of the internals of our components, we can get surprisingly good results with the information we do have.

For the impatient reader, the results are summarized in section 8.

1. Quick Review of Matrix Optics

It is not our purpose to provide a detailed course on matrix optics, so we summarize the salient features.

1.1. **Assumptions.** We make the following assumptions:

(1) Geometric Optics: We consider systems whose properties may be derived through tracing rays, while ignoring the wave nature of light.
(2) Paraxial Approximation (First order optics): The source, optical components, and image plane lie along and are perpendicular to an axis. Rays make a small angle to this axis. We only consider terms linear in this angle.
(3) Gaussian Optics: All surfaces are flat or spherical in the region under consideration. We assume that the point on any lens that a ray hits is at a small enough angle to the axis that we need only consider linear or quadratic terms in it[1]. Note that aberrations are ignored, although much of the internal complexity of a camera lens is designed to compensate for them.
(4) Everything is rotationally symmetric about the optical axis.
(5) The ambient medium is air[2], which we take to have index of refraction $n = 1$. The only rays we need consider are through air because any propagation internal to a lens will already have been accommodated in the corresponding matrix.

[1]This angle determines the point of intersection with the lens, not to be confused with the angle of the incoming ray relative to the x-axis.
[2]Though the internals of a component may not have unit n, and the effective index of refraction for a composite system may not be 1.

1.2. **An Important Optics Equation.** In elementary optics we learn that, for any lens, the condition to achieve focus is

$$\frac{1}{f} = \frac{1}{s_1} + \frac{1}{s_2} \tag{1}$$

where f is the focal length, s_1 is the distance from the image to the lens and s_2 is the distance from the lens to the object. We assume s_1 and s_2 to be positive, though they can become negative if one starts dealing with virtual images or negative focal lengths. As will be discussed, for a thick lens s_1 and s_2 are measured to and from the respective principal planes. For a thin lens, those planes coincide at the lens location.

1.3. **Basic Idea of Matrix Optics.** Matrix optics provides a simple means of calculating the effect of successive components in an optical path. If we think of the x-axis as our center line, then each component sits along it and perpendicular to it. Rays come in at a (small) angle from the left and emerge on the right. The utility of matrix optics derives from our ability to consider each component as a matrix and simply multiply them to get a composite matrix that represents the aggregate system. The matrices themselves are transforms in a vector space[3].

1.3.1. *Ray Vector.* Each light ray is traced from an object on the left to an image on the right. Because of our assumptions, we need only consider the progress of a given ray in two dimensions. At each point the ray is represented by two values

$$\begin{pmatrix} y \\ \alpha \end{pmatrix}$$

This vector represents the ray of light as it passes a given location x. The component y is the location of the ray on the y-axis, while α is its angle to the x-axis. The sign of y is taken be positive if above the x-axis and negative if below it. The sign of α is taken to be positive if the ray is moving up to the right and negative if down to the right.

In describing locations on the x-axis, we refer to points and planes interchangeably where unambiguous. The latter are perpendicular to the axis.

1.3.2. *Optical Component.* In our terminology, an optical component is an interval on the x-axis along with the exact configuration of glass and air that the ray would pass through in traversing that interval. We treat this as a black box that can be moved around if needed, as long as the relative positions of the internal elements relative to the boundary remain the same. For example, component **A** could be defined as spanning the interval $[x_1, x_2]$ with specific thin lenses located at $x_a, x_b \in [x_1, x_2]$. That is, a ray would propagate from x_1 to x_a in air, then through the thin lens at x_a then between x_a and x_b in air, through the thin lens at x_b, and then from x_b to x_2 in air. We could move **A** to any position and it will span an interval of length $x_2 - x_1$ with lenses at distances $(x_a - x_1)$ and $(x_b - x_1)$ relative to the new front point. Put simply, it can be moved as a unit. However if the internal positions of the lenses change or we change the air intervals on either end (thus increasing or decreasing the overall size) or add or subtract lenses, then the component itself has changed. Note that a fixed length of air is considered an optical component from our standpoint.

[3]There are several ”matrix methods” in optics. What we refer to here are called ”ray transfer” or ”ABCD” matrices.

As an aside, it is irrelevant whether we use open or closed intervals[4]. Items like thin lenses or single surfaces with no extent can be included without problem.

1.3.3. *Matrix M.* An optical component[5] **A** has a fixed matrix M_A associated with it. The latter describes the transformation of any ray that propagates through the component. If the component spans the interval $[x_1, x_2]$ then

$$\begin{pmatrix} y \\ \alpha \end{pmatrix}_{x_2} = M \cdot \begin{pmatrix} y \\ \alpha \end{pmatrix}_{x_1}$$

Note that we need not know effective lens positions or cardinal points or anything else. Even the length of the component is irrelevant[6].

1.3.4. *Constraint on M.* Any 2-dimensional matrix can be written

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

However, there is a constraint on our optical matrices. When the ambient medium is the same on both ends, conservation of energy dictates that

$$\det M = 1$$

All the matrices must be unitary; therefore each has three degrees of freedom. From this point on, all matrices are assumed to be unitary unless otherwise specified.

1.3.5. *Units.* It isn't a problem that y and α have different units. We just need to make sure that our matrices manage units accordingly. Specifically, if $[L]$ represents a length unit and '$-$' denotes a dimensionless quantity, our vector and matrix elements have the following units:

$$v : \begin{pmatrix} [L] \\ - \end{pmatrix}$$

$$M : \begin{pmatrix} - & [L] \\ [L^{-1}] & - \end{pmatrix}$$

1.3.6. *Alternate Convention.* Some authors use an alternate convention, defining their ray vector as

$$\begin{pmatrix} \alpha \\ y \end{pmatrix}$$

In that case, we must swap the elements B and C in our matrix to get

[4] We're not attempting to be mathematically rigorous.

[5] The component must satisfy out linearity assumptions, as stated earlier.

[6] It implicitly appears in M through whatever air intervals are included.

$$\begin{pmatrix} \alpha' \\ y' \end{pmatrix}_{x_2} = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ y \end{pmatrix}_{x_1}$$

1.4. **Combining Matrices.** If we consider a ray as passing the plane at x_1 with value $\begin{pmatrix} y \\ \alpha \end{pmatrix}_{x_1}$, propagating through a sequence of n components, and passing the plane x_2 with value $\begin{pmatrix} y \\ \alpha \end{pmatrix}_{x_2}$, then its transformation is given by

$$\begin{pmatrix} y \\ \alpha \end{pmatrix}_{x_2} = M_n \cdots M_1 \begin{pmatrix} y \\ \alpha \end{pmatrix}_{x_1}$$

where $M_1 \cdots M_n$ is the set of matrices associated with the components in the order they are traversed. That is, M_1 is the matrix of the component first encountered by the ray.

1.5. **Object-Image Path.** Most often, we consider a ray emerging from an object on the left, propagating through a series of components, and converging to an image plane on the right. In this case, the entire optical path is represented in the matrices. The latter must include the propagation from the object to the first component and from the last to the image plane. This is useful when determining the requirements for an image to be in focus.

1.6. **Choice of Matrices.** As discussed, a component represents a "black-box" that has a specific length and configuration of internal air and glass. This is what the associated matrix represents. When presented with an interval and a sequence of air and lenses, how do we define our components? We may partition the interval from object to image in any way we choose. Our only constraint is that every point be represented in one and only one component (and matrix).

Matrix multiplication is associative, and it is clear that this approach is consistent. We may combine components or separate them into sub-components at will as long as the result is a proper partition of the optical path.

What determines our choice of partition? There are two factors:

(1) Certain matrices are easy to derive. Specifically, we easily can deal with thin lenses, surfaces, air-gaps, and thick-lenses.
(2) There may be a natural breakdown. For example, we shall deduce the matrices for various camera components. Each has a specific boundary and unknown internal structure. The former may change extent through focusing or zooming, but at any given setting is of fixed extent.

In most cases, there is an obviously preferable breakdown of the path into intervals.

1.7. **Sign Conventions for Distance.** The sign conventions regarding distances can get fairly involved (particularly when virtual images come into play), so we keep direction and length separate to simplify things. Unless otherwise stated we mean a positive quantity when we refer to a distance, and our use of signs reflects this.

1.8. **Reversed Matrix.** To find the matrix for a reversed component, we note that it satisfies:

$$\begin{pmatrix} y' \\ \alpha' \end{pmatrix} = M \begin{pmatrix} y \\ \alpha \end{pmatrix}$$

$$\begin{pmatrix} y \\ -\alpha \end{pmatrix} = M_{rev} \begin{pmatrix} y' \\ -\alpha' \end{pmatrix}$$

The minus signs arise from the reversal of direction of the rays; the entry and exit points simply exchange places but the angle reverses itself when considering the ray as entering the lens instead of exiting.

Recall that for a unitary 2×2 matrix M, the inverse is[7].

$$M^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$$

Defining a non-unitary angle-reversal matrix

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

we can rewrite our equation as

$$RM_{rev}RM = I$$

Although R isn't unitary, the presence of two R's restores unitarity to the equation. This can be rewritten (noting that $R^{-1} = R$) as

$$M_{rev} = RM^{-1}R$$

which yields

$$M_{rev} = \begin{pmatrix} D & B \\ C & A \end{pmatrix}$$

Reversing a component results in a simple exchange of A and D in the corresponding matrix.

1.9. **Powers and Focal Lengths.** For notational simplicity, we use powers instead of focal lengths in the matrices. The "power" of a lens is defined as[8]

$$\phi = \frac{1}{f}$$

[7]For a general 2×2 matrix it is this divided by $\det M$.

[8]Actually, it is $\frac{n}{f}$ but in our case $n = 1$.

1.10. **Thin Lens Matrix.** A thin lens embodies the refractive effect of two surfaces with no thickness in between. That is, the effect on the ray is as if it crossed each of those surfaces without spending any time inside the glass. As a result, the ray is bent but not displaced.

The associated matrix is

$$M_{thin} = \begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}$$

where $\phi = \frac{1}{f}$ is the power of the lens. f can be derived from properties of the two surfaces using the lensmakers equation, which we won't discuss here.

1.11. **Space Matrix.** Every piece of the ray's path must be represented in some matrix. This means we must include a matrix for any space between components. Note that this "space" corresponds to the interval between the end of one component and the beginning of another

$$M_{space} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

where t is the relevant distance along the x-axis. A space matrix displaces but does not bend the ray.

1.12. **Thick Lens Matrix.** A thick lens has two refractive surfaces with glass of nonzero thickness in between. Unlike with the thin lens, we must account for internal propagation. All real lenses are thick lenses, though we approximate some as thin for simplicity. If the powers of the individual surfaces are ϕ_1 and ϕ_2, the distance between their vertices is t, and the index of refraction of the glass is n, then (defining the effective length $\tau = \frac{t}{n}$), the appropriate matrix is

$$M_{thick} = \begin{pmatrix} 1 - \phi_1\tau & \tau \\ \phi_1\phi_2\tau - \phi_1 - \phi_2 & 1 - \phi_2\tau \end{pmatrix}$$

where ϕ_1 and ϕ_2 are the effective powers of the two surfaces and τ is the distance between the vertices. A possible meaning will be introduced shortly.

It is not hard to demonstrate that the focal length of a thick lens in air is the same on both sides and has the value

$$f = \frac{1}{\phi_1 + \phi_2 - \phi_1\phi_2\tau}$$

However, it is measured from the Principal Planes rather than any common center point.

1.12.1. *Degrees of Freedom.* In first order optics, the properties of a thick lens are completely determined by three parameters: ϕ_1, ϕ_2, and τ. This is the same number of degrees of freedom as in a 2x2 unitary matrix. We may interpret every such matrix as an effective thick lens, though the utility of doing so depends on the the context. This one-to-one correspondence is easy to construct from the matrix above, and we will make use of it shortly.

1.12.2. *Focal Length of a Matrix.* We shortly will derive the optical properties of the effective thick lens associated with a matrix. For now, we simply observe that its focal length in air is

$$f_{eff} = \frac{-1}{C}$$

1.12.3. *Important Points and Planes.* There are a number of important planes (or points on the x-axis) associated with a thick lens. Their locations for the effective thick lens of a system are of general interest, and provide insight into its optical behavior.

- O and I are the object and image points.
- F and R are the front and rear physical boundaries of the interval represented by the matrix. They also are known as the vertices.
- P_f and P_r are the principal planes.
- F_f and F_r are the focal planes.

The points F_f, F_r, P_f, and P_r are four of the six ”Cardinal Points” of the lens[9].

1.12.4. *Vertices.* The vertices of a thick lens are the points at which the outer surfaces intersect the x-axis. They are the extent of the actual glass. For an effective thick lens, the vertices correspond to the endpoints of the interval represented by the relevant matrix. For a thin lens the vertices coincide.

1.12.5. *Principal Planes.* The effect of a thick lens on light can be thought of as that of a thin lens of the same focal length but with a gap in space. This defines the principal planes, P_f and P_r. Let us denote the effective thin lens as T. Light acts as if it enters T at the plane P_f and emerges with the same y' and α' as it would were T a true thin lens – but instead of doing so at P_f, it does so at P_r. That is, the ray magically appears at P_r. Equivalently, we can treat the optical path as if it had a thin lens at P_f and the interval $[P_f, P_r]$ were removed. For a thin lens, the Principal Planes coincide with the vertex.

1.12.6. *Focal Planes.* In air, the focal length is the same on each side of a lens. However, it is measured from the principal planes. That is, the focal planes are at $P_f - f$ and $P_r + f$. Likewise, equation 1 holds, but with s_1 and s_2 measured from the principal planes to the sensor and object. As mentioned, the region $[P_f, P_r]$ is treated as if it does not exist. For a thin lens, the Focal Planes are symmetrically located around the vertex.

1.13. **Two Thin Lenses a Thick Lens Make.** It is fairly easy to see that two thin lenses with a separation of τ are equivalent to a thick lens. Specifically, if

$$M_{thin1} = \begin{pmatrix} 1 & 0 \\ -\phi_1 & 1 \end{pmatrix}$$
$$M_{thin2} = \begin{pmatrix} 1 & 0 \\ -\phi_2 & 1 \end{pmatrix}$$
$$M_{gap} = \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix}$$

Then

[9] The other two are called Nodal Points and don’t concern us here.

$$M_{thin2}M_{gap}M_{thin1} = \begin{pmatrix} 1-\phi_1\tau & \tau \\ \phi_1\phi_2\tau - \phi_1 - \phi_2 & 1-\phi_2\tau \end{pmatrix}$$

This is just the matrix for a thick lens, and provides an easy interpretation of the latter's elements. Specifically, we may read off C to get the usual equation for combining thin lens focal lengths:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{\tau}{f_1 f_2}$$

Conversely, we may regard a thick lens as two thin lenses with a gap. Given a thick lens

$$M_{thick} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

we extract

- $\phi_1 = \frac{1-A}{B}$
- $\phi_2 = \frac{1-D}{B}$
- $\tau = B$

It is easy to see that the thin lenses do not sit at the principal planes of the corresponding thick lens. Perhaps it is less obvious that they do not sit at the vertices either. This is because of the internal index of refraction of a thick lens. When discussing degrees of freedom, we noted that ϕ_1, ϕ_2, and τ completely determine the optical behavior of the lens. This is true as far as ray-tracing goes, and there is indeed a one-to-one correspondence between optical matrices and sets of these three parameters. If we start asking questions about other optical quantities which do not directly affect the behavior of rays, then we may require more information. For example, many internal configurations of optical components yield the same matrix for a given interval. The resulting focal length and principal planes must be the same, but the vertices need not. They represent extraneous information about the internal structure. If we also vary the interval covered by the matrix, as we must when considering a thick lens comparison, the principal planes change as well[10]. Consider a thick lens with internal index of refraction n. We did not derive the thick lens matrix in section 1.12, but we noted that $\tau = \frac{t}{n}$. Because the lens is made of glass, the distance is increased by a factor of n. This does not affect the optical properties of the component as long as the outside medium remains air. However, when attempting to compare a thick lens with two thin lenses the distinction becomes important. Two thin lenses separated by distance τ are equivalent to a thick lens of any index of refraction n with the same matrix elements and a distance between vertices of $n\tau$. As n typically is in the range of 1.5 for glass, the difference can be quite significant. Note that the comparison involves the same matrix but different intervals. If we require that the interval covered by the matrix be constant then we must add air gaps outside the thin lenses and adjust our thick lens accordingly.

1.14. **Two Thick Lenses a Thick Lens Make, BUT...** Similarly, we could combine two thick lenses with a gap between them. The resulting calculation is rather cumbersome and tangential to our current discussion. We simply note that the problems with comparison that arise from the case of two thin lenses are significantly uglier.

[10]Obviously, the focal length does not.

2. Camera Component ↔ Matrix

2.1. **Optical Parameters.** Our purpose is to determine the optical properties of a stack of camera components. Let us begin by defining various optical parameters.

2.1.1. *Front and Rear of Camera Lens.* As a point of terminology, the ”front” of the lens is the business end while the ”rear” is that which ordinarily connects to the camera. This is consistent with our use of ”front” and ”rear” in reference to principal planes.

2.1.2. *δ - Distance from Sensor to Rear of Lens.* For many purposes, we require knowledge of the distance between the camera sensor or film and the rear of an attached lens. We denote this δ. It is constant within a camera family, and all compatible lenses are designed to accommodate it. As δ is important for various practical applications, it is reported by camera manufacturers. The two most common values are:

- Canon EOS: $\delta = 44mm$.
- Nikon F: $\delta = 46.5mm$.

As a side note, most cameras indicate the position of the image plane with a marking on the camera body.

2.1.3. *Image Orientation.* Though not an optical parameter per se, the image orientation is of importance. The image on the sensor plane is inverted. The camera compensates for this through the orientation and interpretation of the sensor, resulting in an upright final image. For optical calculations we must remain cognizant of the inversion.

2.1.4. *w_s - Sensor Width.* Although unnecessary for most optical calculations, we sometimes need to know the size of the sensor. For example, in deducing various optical parameters for a lens from empirical measurements the sensor size comes into play. Also it allows us to compute the field of view and adjust for perceived focal length and aperture. Unless otherwise stated, we take the sensor width to be measured horizontally. The sensor dimensions are constant within a class of camera. For example, all Canon EOS cameras have a specific δ but the w_s differs between their full-frame and APS-C cameras. Some common values are:

- Full-frame (or 35mm): 36 x 24 mm.
- Canon APS-C: 22.2 x 14.8 mm.
- Nikon APS-C: 23.6 x 15.7 mm.
- Four-Thirds: 17.3 x 13 mm.

2.1.5. *L - Length of System.* For certain purposes[11], we require knowledge of the physical length of a lens or stack. We define this to be L, and it is measured from the rear seal with the camera to the frontmost point. A few words of caution in its use:

- It can change as we adjust the focus or zoom setting of a lens.
- The reported value usually is the maximum across focus/zoom settings.
- When using a reported value it is important to know whether it is measured from the end of the canister, the frontmost vertex of the glass, or the point where a filter would sit. Most likely it is the last. It is okay for us to use any choice as long as we are consistent. We simply must amend our calculations accordingly.

[11] Such as the determination of cardinal points, or conversions between various object distance conventions.

- It is best to measure L while the lens is attached to the camera. Otherwise the protruding bayonet mount may make it difficult to locate the appropriate rear point.
- When measuring a lens, we must make sure that any filters or attachments are removed. Many photographers use a protective glass or UV filter, and this must not be included in the length.

2.1.6. d_{fo}, d_{so}, $\tilde{d}_{fo}$, $\tilde{d}_{so}$ - *Distance to Object.* The minimum distance to an object is a quantity that often is available for camera lenses. It easily may be measured as well, and is of interest as a constraint in macrophography. First, let us clarify what it means. As embodied in equation 1, associated with any object distance s_2 from the front principal plane of a system there is an image at some distance s_1 from the rear principal plane. However, s_1 and s_2 are not of direct interest to us because we rarely know the location of the principal planes a priori. Nor are they useful reference points for a comparison of different systems. Rather, there are several possible definitions that may be of use, only three of which make sense for comparison across systems. We could measure:

(1) From the front of the physical unit. If the unit changes size, the reference plane will move with the front point.
(2) From the rear of the lens unit (that is, the seal with the camera).
(3) From the image plane, the location of the sensor.

The latter two choices differ by δ, which is constant for a camera family. From an optics standpoint, the rear of the lens unit isn't special. We stick with the first and third choices, and define two corresponding distances. In certain cases, such as focal play or zooming, these quantities are variable[12], and it is useful to consider the minimum focal distance across settings. This is a property of the lens and plays a significant role in our analysis. We define the following:

- d_{fo} is the distance from the frontmost physical point in the system to the object.
- d_{so} is the distance from the sensor plane to the object.
- $\tilde{d}_{fo}$ is the minimum value of d_{fo} for which focus may be achieved when there is a given range of settings under consideration.
- $\tilde{d}_{so}$ is the minimum value of d_{so} for which focus may be achieved when there is a given range of settings under consideration.

If the system length at the relevant settings is L, then $d_{so} = d_{fo} + L + \delta$. Note that L may vary as we focus, and in theory the same does *not* hold for $\tilde{d}_{so}$ and $\tilde{d}_{fo}$. The minima may be at different values of L.

As photographers we generally care about $\tilde{d}_{fo}$ as it determines how close we can get to a subject for macrophotography. However, the reported minimum object distance for a lens usually is $\tilde{d}_{so}$. Thus when Canon tells us that their 70-200mm f/4L has a minimum object distance of 1.2m, that is measured from the sensor plane.

Note that from a matrix standpoint, we may use either method as long as the matrices are chosen accordingly. Also note that some values of d_{fo} or d_{so} may not be accessible because the associated s_1 may lay outside the range of focal play. This is what defines $\tilde{d}_{fo}$ and $\tilde{d}_{so}$. However when considering a stack of components, we must allow all possible values (even virtual objects) because the overall combination may result in a reasonable image even if the individual lenses do not. For example, it often makes sense to reverse one lens onto another well within the $\tilde{d}_{so}$ of either.

[12] Strictly speaking, our definition of "component" corresponds to a specific choice of focus and zoom. We really must consider a set of components when we speak of a range. However we will be cavalier in our terminology, and the context will be clear.

2.1.7. *m_o, m_u - Magnification.* One of the most important optical parameters is the magnification. A major purpose of stacking is to achieve very high magnifications. While the magnification is a function of object distance (however we measure it), in most cases we only care about the maximum magnification. Unless otherwise specified, this is what we refer to.

Before proceeding, we should briefly mention one possible point of confusion. The term ”magnification” can be used in a number of ways. First, let us list some common ways in which it is reported:

- $n : m$ means that the image is $\frac{n}{m}$ times life-size. Usually, either n or m is chosen to be 1. Thus, $5 : 1$ means 5 times life-size (a macro shot), while $1 : 4$ means a quarter of life-size.
- n or nX means n times life-size. Thus $5 : 1$ and $1 : 0.2$ and $5X$ all mean the same thing.

The differences in meaning arise from the definition of ”life-size”. This may be an absolute reference size, or something related to the camera. The two most common definitions are:

- Literal: The image on the sensor or film is n times larger than the corresponding object in life. Thus, a 1 inch wide object at 0.5 magnification would occupy 0.5 inches of the sensor. Note that this does not depend on the size of the sensor. Within a sign, it is the mathematical magnification $m = \frac{y'}{y}$ that appears in calculations.
- Print: Often, a 6”x4” print is taken as a fixed reference size. At 2:1 magnification, a 1 inch wide object would appear 2 inches wide on the print. It thus must consume $\frac{1}{3}$ of the width of the sensor or film from which the print is produced.

The translation between the two definitions depends on the size of the sensor. Denoting the sensor width w_s, let us define η to be the conversion factor $\eta = \frac{w_s}{152.4}$ for a 6”x4” reference print. Two common values are[13]:

- Full-frame 35mm camera: $w_s = 36$ and $\eta = 0.23622$.
- APS-C 1.6X cropped sensor camera: $w_s = 22.2$ and $\eta = 0.14566$.

To obtain the print magnification, we divide the literal magnification by η. For example, a 0.4X literal magnification corresponds to a 1.7:1 print magnification if using a full-frame sensor or a 2.75:1 print magnification if using 1.6X cropped sensor[14]. Similarly, Canon quotes a 5x (or 5:1) magnification for their MP-E65 lens. This too is relative to sensor size (it assumes full-frame), so it translates to 21:1 in print terms.

For our purposes, all magnifications will be literal within a sign. The latter bears some consideration. A camera lens produces an inverted image on the focal plane. Sometimes we wish to account for this and sometimes we do not. Thus we define two quantities (both literal magnifications):

- m_o: The optical magnification. It is the quantity that appears in calculations and is negative for a camera lens. It is defined as $m_o = \frac{y_i}{y_o}$ where y_i and y_o are the ray heights at the object and image planes. Equivalently, $m_o = \frac{s_1}{s_2}$.
- m_u: This is the magnification as seen in a photo. It is defined as $m_u = -m_o$ to account for the internal inversion within the camera. It almost always is positive for a camera lens, but could prove negative for a stack of components (meaning that $m_o > 0$ and the final image is upside-down). Both the reported and relevant deduced maximum magnifications are m_u values.

[13] For our purposes, we ignore any difference in aspect ratio between the sensor and the print.

[14] This is larger not because there is more object on the sensor, but rather less sensor converting to the same print size.

When considering focal or zoom ranges, both m_u and m_o are taken to be extreme values unless otherwise specified. Note that these need not be obtained at the minimum object distance.

3. Matrix → Lens

Though it may not always be useful to do so, we can extract the optical parameters of the effective thick lens asssociated with an arbitrary optical matrix. We assume knowledge of δ and w_s. Let us suppose that we are handed a matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $\det M = 1$.

3.1. The Interval Associated with M.

Note that M isn't associated with an interval per-se, even if the component matrices used to construct it were deduced from specific intervals. A given M could correspond to any interval. This carries two degrees of freedom. First there is the location of the interval, a degree of freedom in the definition of "component" as well. Second there is the length of the interval. Though the vertex locations relative to their neighboring principal planes are fixed, the distance between the latter is not. Recall that it is invisible from a physical standpoint. Without knowledge of L, we cannot deduce it.

Therefore, in deriving optical parameters the placement of the effective lens associated with a matrix is of importance. In our case, we wish to consider the matrix for a stack of components. We therefore choose the rear vertex to be at the seal with the camera and the front vertex to be a distance d_{fo} from the object. Note that there is a fixed value of d_{fo} in this approach. To defined $\tilde{d}_{fo}$ we need a range. This corresponds to the play in settings (focal or zoom) of all the components in the stack, which in turn is associated with a range of matrices – though we sometimes can make simplifying assumptions. We will discuss this later. For now, we note that a single matrix M corresponds to a single object distance d_{fo} when we lock the image plane at δ behind the rear vertex of the unit. Note that if we wish for d_{so} instead of d_{fo} we need to know L as well.

3.2. f - The focal length.

In air this is the same on both sides, and is measured from the principal planes. It can be read off of the matrix as

$$f = -\frac{1}{C}$$

3.3. d_{fo} - Finite case.

As mentioned, we ignore focal play and assume a fixed object distance d_{fo}. Let us first assume that d_{fo} is finite and determine both it and m_o from a given matrix.

Our optical path consists of light from the object traveling to the front of the lens, through the lens, and then from the rear of the lens to the sensor. The corresponding matrices are

$$M_{tosensor} = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix}$$

$$M_{lens} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$M_{fromobject} = \begin{pmatrix} 1 & d_{fo} \\ 0 & 1 \end{pmatrix}$$

and the ray transformation is given by

$$\begin{pmatrix} y' \\ \alpha \end{pmatrix} = M_{tosensor} M_{lens} M_{fromobj} \begin{pmatrix} y \\ \alpha \end{pmatrix}$$

A simple calculation yields

$$y' = y(A + \delta C) + \alpha(d_{fo}A + B + \delta(d_{fo}C + D))$$
$$\alpha' = y \cdot C + \alpha(d_{fo}C + D)$$

To achieve focus, all rays from a fixed y on the object must converge to a fixed y' on the sensor. So, we need y' to be independent of α. This means

$$d_{fo}A + B + \delta(d_{fo}C + D) = 0$$

which yields

$$d_{fo} = -(\frac{B + \delta D}{A + \delta C})$$

3.4. m_o **and** m_u **at finite distance** d_{fo}**.** Following our calculation of d_{fo}, we immediately can compute $m_o = \frac{y'}{y}$ at that object distance:

$$m_o = \big(A + \delta C\big)$$

.

Similarly,

$$m_u = -\big(A + \delta C\big)$$

.

3.5. **Deferred - Infinite** d_{fo}**.** We will return to the case of infinite object distance in section 4.4. This limit is best introduced from the other direction, first computing the matrix associated with the lens. For now we simply note that, as expected from the finite result, if $A + \delta C = 0$ the inferred lens should have infinite d_{fo} (and d_{so}).

3.6. **Thick Lens Locations from Matrix.** If we are given a matrix M and the length L of the interval associated with it, we may compute the locations of various planes in the corresponding effective thick lens. Note that L for a stack is simply the sum of the lengths of the underlying components.

- Image Sensor: We define this to be $x_I = \delta$. This places our origin at the rear of the physical system.

- Rear of Physical Unit (rear vertex of effective thick lens): $x_{vr} = 0$ by choice.
- Front of Physical Unit (front vertex of effective thick lens): $x_{vf} = -L$.
- Object: The object is located at $x_o = -d_{fo} - L = \left[\frac{B+\delta D}{A+\delta C}\right] - L$.
- Rear Principal Plane: $x_{pr} = \frac{1-A}{C}$.
- Front Principal Plane: $x_{pf} = -L - \frac{1-D}{C}$.
- Rear Focal Point: $x_{fr} = \frac{-A}{C}$.
- Front Focal Point: $x_{ff} = -L + \frac{D}{C}$.

The relevant distances are:

- Distance from sensor to rear principal plane: $s_1 = \delta + \left(\frac{A-1}{C}\right)$
- Distance from second principal plane to object: $s_2 = d_{fo} + \left(\frac{D-1}{C}\right) = \frac{1-A-\delta C}{C(A+\delta C)}$.
- Distance between principal planes: $L + \left(\frac{2-A-D}{C}\right)$

4. Lens → Matrix

4.1. **Obtaining the Optical Parameters.** To determine the optical characteristics of a stack of photographic equipment, we first must obtain the matrices associated with the components. Unfortunately, manufacturers do not provide such information directly. However it generally can be deduced from the parameters they do provide and/or some simple measurements. All lenses list f on their canister, and many include d_{so} as well.

For Canon, a list currently can be had here (though it is best to use the parameters listed in each lens' manual). `''http://www.usa.canon.com/app/pdf/lens/EFLensChart.pdf''`

An unofficial set of Nikon data can be found here: `''http://grwsystems.net/Nikon/index.html''`

As before, let us first consider a lens with a specific focal plane (i.e. no focal play).

4.2. **Finite Distance, Using m_u, $\tilde{d}_{so}$, L, and f.** The parameters that are easiest to obtain are the focal length f, maximum upright magnification m_u, and minimum object distance (from the sensor) $\tilde{d}_{so}$ at which focus can be achieved. While it is theoretically possible[15] that a complex lens attains the maximum magnification for some $d_{so} > \tilde{d}_{so}$, it is unlikely and we assume that the two coincide. We also assume that $\tilde{d}_{fo}$ and $\tilde{d}_{so}$ coincide, though they too could occur at different lens settings[16]. Because $\tilde{d}_{so}$ is reported, we need the length of the unit L. We assume the reported L is at $\tilde{d}_{so}$. In summary, we assume that the maximum unit size, the minimum object distances (from the sensor and from the front) and the maximum magnification all coincide. In the rare case where a lens is not maximally extended at near focus, it may be necessary to compensate by measuring the difference.

From these parameters and the requirement that $\det M = 1$, we can deduce the matrix M for the lens. Specifically, we assume knowledge of the parameters f, L, $\tilde{d}_{so}$, and m_u, and that $\tilde{d}_{fo} = \tilde{d}_{so} - L - \delta$ and m_u coincides with the stated $\tilde{d}_{so}$. Recall that $m_o = -m_u$.

4.2.1. *Matrix in terms of f, $\tilde{d}_{fo}$, and m_u.* The calculation is exactly the inverse of that performed in section 3.3. Because that calculation was performed in terms of f, $\tilde{d}_{fo}$, and m_o, let us first state the results using these.

[15] For example, if the principal planes shift in a weird way inside the canister.

[16] For example, if L changes while focusing or zooming.

(2)
$$\begin{aligned} A &= \frac{\delta}{f} + m_o \\ B &= -m_o \cdot \tilde{d}_{fo} - \frac{\delta}{m_o} - \frac{\delta \tilde{d}_{fo}}{f} \\ C &= -\frac{1}{f} \\ D &= \frac{\tilde{d}_{fo}}{f} + \frac{1}{m_o} \end{aligned}$$

Or as a matrix:

$$M = \begin{pmatrix} \frac{\delta}{f} + m_o & [-m_o \cdot \tilde{d}_{fo} - \frac{\delta}{m_o} - \frac{\delta \tilde{d}_{fo}}{f}] \\ -\frac{1}{f} & \frac{\tilde{d}_{fo}}{f} + \frac{1}{m_o} \end{pmatrix}$$

4.2.2. *Matrix in terms of f, $\tilde{d}_{so}$, L, and m_u.* We also can express the results in terms of the exact quantities we are given:

(3)
$$\begin{aligned} A &= \frac{\delta}{f} - m_u \\ B &= m_u \cdot (\tilde{d}_{so} - L - \delta) + \frac{\delta}{m_u} - \frac{\delta(\tilde{d}_{so} - L - \delta)}{f} \\ C &= -\frac{1}{f} \\ D &= \frac{(\tilde{d}_{so} - L - \delta)}{f} - \frac{1}{m_u} \end{aligned}$$

Or as a matrix:

$$M = \begin{pmatrix} \frac{\delta}{f} - m_u & [m_u \cdot (\tilde{d}_{so} - L - \delta) + \frac{\delta}{m_u} - \frac{\delta(\tilde{d}_{so} - L - \delta)}{f}] \\ -\frac{1}{f} & \frac{(\tilde{d}_{so} - L - \delta)}{f} - \frac{1}{m_u} \end{pmatrix}$$

4.3. **Infinite Distance.** When the object distance is infinite, the problem is a little more challenging.

For infinite or very large d_o, the linear magnification is not useful[17]. A more salient characteristic is the field of view, which we denote γ. This is the physical angle subtended by an object that completely fills the image[18]. The rays from a distant object are parallel at first order, and the image forms on the rear focal plane.

[17] For afocal systems such as telescopes, we instead compare the relative widths of parallel entry and exit beams to obtain an angular magnification.

[18] Horizontally, vertically, or diagonally as specified.

4.3.1. *Field of View.* To compute the field of view we must know the physical width of the sensor in the direction of the angle we are measuring. Let us suppose it to be w_s. Then

$$\gamma = 2tan^{-1}\frac{w_s}{2f}$$

For $w_s << f$ (which is not embodied in our other small-angle assumptions), this may be approximated as

$$\gamma = \frac{w_s}{f}$$

4.3.2. *Degrees of Freedom.* Unfortunately, γ is a function of f and won't serve as a replacement for m as far as information content. We know f, but the object distance is infinite and the magnification is zero. Even with the unitarity condition, we can lock down only 3 elements in M. An additional assumption or measurement is needed.

We take the simplest route and assume that the effective lens moves in tandem to focus. That is, we assume that the process of focusing involves shifting an effective lens within the canister. Most likely, the focusing mechanism is more complex and involves changes to all three of A, B, and D. However, in the absence of additional information this is the most reasonable choice.

To focus the effective lens, we shift its distance as a whole from the sensor[19]. Because the object distance will become infinite, it is irrelevant whether we use d_{fo} or d_{so}. We arbitrarily choose d_{fo} here.

Our starting point is a matrix derived from the lens at finite object distance. This could be any object distance d_{fo} at which we know m_u, but we are most likely to have that information at the minimum object distance. Given a matrix

$$M_{fin} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

at that object distance, we simply reduce the gap on one side and increase it on the other. Let us do so by some arbitrary amount t.

$$M' = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$$

4.3.3. *M_{inf} in terms of M_{fin}.* Expanding M', we get

$$M' = \begin{pmatrix} A + tC & [B - tA + tD - t^2C] \\ C & D - tC \end{pmatrix}$$

Denoting by d_{fo} and m_u the object distance and magnification associated with M_{fin}, and by d'_{fo} and m'_u the object distance and magnification associated with M', it is not hard to see that

[19]Consequently, the principal planes move in parallel.

$$m'_u = m_u - tC$$

$$f' = f$$

$$d'_{fo} = \frac{tm_u - t^2C + d_o m_u + tD}{m_u - tC}$$

To achieve focus at $d_{fo} \to \infty$, we require that $t \to \frac{m_u}{C} = -(\delta + \frac{A}{C})$. Substituting this in, we see that

$$M_{inf} = \begin{pmatrix} -\delta C & \left[B - \frac{(A+\delta C)(D+\delta C)}{C}\right] \\ C & D + A + \delta C \end{pmatrix}$$

4.3.4. *M_{inf} in terms of f and m_u at some d_{fo}.* Alternately, if we wish to cast this in terms of f and the value of m_u at some finite d_{fo}, we could use the results of section 4.2 to get

(4)
$$M_{inf} = \begin{pmatrix} \frac{\delta}{f} & [f + m_u\delta + \frac{\delta}{m_u} - \frac{\delta d_{fo}}{f}] \\ \frac{-1}{f} & \frac{d_{fo}}{f} - m_u - \frac{1}{m_u} \end{pmatrix}$$

4.3.5. *M_{inf} in terms of f, m_u, L, and $\tilde{d}_{so}$.* Lastly, if we wish to use the reported values for the near focus end of a lens, We get

(5)
$$M_{inf} = \begin{pmatrix} \frac{\delta}{f} & [f + m_u\delta + \frac{\delta}{m_u} - \frac{\delta(\tilde{d}_{so}-L-\delta)}{f}] \\ \frac{-1}{f} & \frac{(\tilde{d}_{so}-L-\delta)}{f} - m_u - \frac{1}{m_u} \end{pmatrix}$$

4.3.6. *Focal Play.* If m_u is the maximum magnification, assumed to be attained at nearest focus (using either $\tilde{d}_{fo}$ or $\tilde{d}_{so}$), then the total focal play can be computed as

$$t_{play} = m_u f$$

This assumes that the lens can achieve focus for an infinitely distant object. That usually is the case with ordinary lenses, but not macro lenses. Also, if our matrix is that of a composite system then the associated effective lens may have a finite maximum object distance even if the component lenses do not. In those cases, we must explicitly determine the focal play from the matrices at the minimum and maximum object distances. As discussed in section 3.6, The motion in the rear focal plane is given by $|\frac{A_2 - A_1}{C}|$. This gives us a focal play of

$$t_{play} = |(m_2 - m_1)f|$$

where m_1 and m_2 are the magnifications at the minimum and maximum object distances (either m_o or m_u will do).

4.4. **Matrix → Lens: Infinite d_{fo}.** We now return to the calculation deferred from section 3. If in computing the values of f, d_{fo}, and m_u associated with a system's matrix M, we run into divergent values, we may need to check for an infinite object distance. If $A = -\delta C$ then the matrix is given by equation 5. If divergent and not consistent with that form then we may not be able to assume parallel focal motion and another assumption (or measurement) is needed.

As d_{fo} is infinite and m_u is 0, the only quantities we need to extract are f and γ. These are easy to obtain:

$$f = -\frac{1}{C}$$

$$\gamma = \frac{w_s}{f}$$

4.5. **Aside: Principal Planes of a Camera Lens.** Where do the principal planes sit for a real camera lens? Often, one of the principal planes is near the front end of the lens but we cannot be certain of this. As a simple example, consider an actual thin lens in a 172mm long canister with a focal length of 200mm and a distance from the rear to the sensor of 44mm. From equation 1, we can figure out the object distance s_2 as a function of the lens location in the canister. At distance x from the rear of the canister, $s_1 = 44 + x$. As s_1 varies from 44 to 216, s_2 goes from -56 to $-\infty$ (at $s_1 = 200$) and then immediately jumps to positive infinity and proceeds down to 2632 at $s_1 = 216$. So, an infinite object distance is obtained when $x = 156$ and $s_1 = 200$. Any value of s_2 that leads to a position within or behind the canister is not useful. The minimum object distance of 2632 is achieved at $s_1 = 216$ and $x = 172$.

In a real lens system it is possible for the effective x to sit outside of the canister. Moreover the focal play is restricted. As a more realistic example, let us consider an effective thick lens for the Canon 70-200mm zoom lens at 200mm focal length. We know that $d_{so} = 1200$ and $m = 0.21$ from the specifications. As before, the canister has length $l = 172$. Consider the space from the sensor to the object to consist of three regions: s_1 is the distance from the sensor to the rear principal plane, s_2 is the distance from the front principal plane to the object, and b is the distance between the principal planes. We know that $m_u = \frac{s_1}{s_2}$, which implies that $s_1 = m_u \cdot s_2$. From equation 1, we have that $f = \frac{m_u \cdot s_2}{1+m_u}$. From these, we obtain

$$s_1 = (1 + m_u)f = 242$$
$$s_2 = (1 + \frac{1}{m_u})f = 1152.38$$

The total distance from sensor to object is $d_{so} = 1200$.

From this we see that

$$b = d_{so} - s_1 - s_2 = -194$$

The principal planes are reversed. This is surprising, but makes sense. In order to have a compact telephoto lens with a short minimum object distance, the rear principal plane must sit forward of the front principal plane. Specifically, the rear principal plane is 252mm in front of the sensor, while the front principal plane is 48mm in front of the sensor.

At infinity focus, the rear principal plane obviously is 200mm in front of the sensor and the front principal plane 6mm in front of the sensor. This assumes parallel motion of the planes (which very well may be wrong for a lens designed this way). The focal play is 42mm. That is, a range of 42mm in the position of the effective lens allows focusing over a range from $d_{so} = 1200mm \ldots \infty$.

5. Other Components

5.1. Telextender.

A telextender is a device which multiplies the focal length of any lens by a constant. It is placed between the lens and the camera. Denoting the multiplier x, we have $f_{new} = xf$. For example, for a 1.4X telextender, $x = 1.4$.

Suppose our lens (which itself could be an effective lens for some system) has matrix

$$M_{lens} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Let us construct a new matrix

$$M_{TE} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

such that $M'' = M_{TE} M_{lens}$ has focal length xf for any f. It is not obvious that such a matrix exists, but we will see that it does.

Multiplying, we get $D'' = AC' + CD'$. We therefore require that $AC' + CD' = \frac{C}{x}$ and

$$C' = 0$$
$$D' = \frac{1}{x}$$

We next consider the condition that $\det M'' = 1$. This requires that $(AA' + CB')\frac{D}{x} - (BA' + DB')\frac{C}{x} = 1$ which means that

$$A' = x$$

This leaves B' as a degree of freedom. However, there is one obvious choice. Given the current values, we can compute the optical parameters

$$(m'_u) = Ax + CB' + \delta xC$$

$$d'_{fo} = -\left[\frac{Bx + DB' + \frac{\delta D}{x}}{Ax + CB' + \frac{\delta C}{x}}\right]$$

$$f' = -\frac{x}{C} = fx$$

Ideally, we would like m'_u and d'_{fo} to be affected in a uniform way across lenses[20]. If we require that $B' + \frac{\delta}{x} = \delta x$ then

$$m'_u = m_u x$$

$$d'_{fo} = d_{fo}$$

$$d'_{so} = d_{so} + L_{TE}$$

where L_{TE} is the length of the telextender unit.

Thus, the matrix for the telextender is:

$$M_{TE} = \begin{pmatrix} x & \delta(x - \frac{1}{x}) \\ 0 & \frac{1}{x} \end{pmatrix}$$

5.2. **Closeup Lens.** A closeup lens reduces the minimum object distance, increasing the maximum magnification. It typically is placed on the front of a lens, much like a filter. We may consider it a thin lens for most purposes[21]. Also, we only know its reported diopter so this is the best approximation we can make. The diopter is just the power ϕ measured in units of inverse meters. Thus a 1.5 diopter lens has $f = 666.66mm$, while a 2.9 diopter lens has $f = 344.84mm$. Using millimeters for everything and denoting by D the reported diopter of the lens, the associated matrix is

$$M_{CL} = \begin{pmatrix} 1 & 0 \\ \frac{-D}{1000} & 1 \end{pmatrix}$$

5.3. **Extension Tube (or adapter with thickness).** An extension tube (or a thick adapter ring) simply requires a space matrix. For a tube of reported thickness t, this is

$$M_{ET} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Note that if the element has a narrow diameter and could potentially block light at that stage in the optical path, then we may need to account for it in our aperture estimate as well.

5.4. **Reversed Lens Matrix.** For a lens sitting in air, the focal distances on both sides should be the same. These are measured from the principal planes, which differ for a thick lens but are the same for a thin lens. It may seem like reversing a lens should have no effect. However, there are several reasons that this isn't quite right[22]:

[20] We're okay with a δ dependence, however, because that is specific to the camera.

[21] Its thickness is nominal, but if the unit takes a lot of space it may be appropriate to apply an M_{space} as well.

[22] We know from section 1.8 that there is an effect.

(1) The focal play has a different effect. Although we haven't discussed it yet, the focal play corresponds to a range of $[s_a, s_b]$ – the distance from the rear principal plane to the camera sensor. Let us assume that the focal mechanism involves a parallel motion of the principal planes. If d_1 is the distance from the rear of the lens to the rear principal plane and d_2 the corresponding distance from the front principal plane to the front of the lens (both taken as positive), when we reverse the lens the effective range becomes $[s_a - d_1 + d_2, s_b - d_1 + d_2]$. Using equation 1, and noting that s_a and s_b represent a range of s_1 values, the range of object distances s_2 at which focus can be achieved has changed.
(2) Although we think of the aperture as the diameter of a lens, in practice it is more complex. For a lens with multiple components[23], the aperture depends on the light path (which varies in width) through many stages. When we reverse the optical path, there is no guarantee of symmetry. Even if there were, the rear of the lens is designed to pass a narrow cone of light to the sensor while the front of the lens is designed to admit as much light as possible. For these reasons, the aperture likely will be significantly reduced. This especially is true of retrofocal wide-angle lenses.
(3) Physically, the components may not fit together. For example, there may be a protruding front piece of glass such as is found in ultra wide-angle lenses.
(4) With a zoom lens the considerations regarding focus are compounded.
(5) Autofocus is lost, though that need not concern us here.

As discussed in section 1.8, the entire effect of lens reversal is to swap A and D in the matrix. Note that this is equivalent to changing $m_u \rightarrow \frac{1}{m_u}$ (and the same for m_o) and $d_{fo} \leftrightarrow \delta$ for a finite object distance. Also note that our result can be applied to a matrix representing any component or combination of components; we need not restrict ourselves to reversing an individual lens.

6. Creating Complex Configurations

When building a system from a sequence of components there are a number of considerations.

6.1. **Focal Play.**

A real lens has focal play to allow it to achieve focus at a range of object distances. In the simplest case, a thin lens is moved so that focus can be achieved. We note from equation 1 that by varying s_1 we can vary s_2. For $s_1 << s_2$, a small range of s_1 values can lead to a wide range of s_2 values. Of course, in a real lens the focusing mechanism may be far more complex. A simple way to incorporate focal play is to create 2 matrices, one at each end of the range. Most lenses have a stated minimum object distance (corresponding to the maximum s_1 in our simple example), and an infinite maximum distance. Any finite maximum distance would be mentioned in the lens specification[24].

It is important to note that even if the individual camera lenses can focus on an infinitely far object, once we start combining matrices the resulting effective lens may not. Worse, the focal length may change. This is easy to see even with thin lenses. Suppose we have two thin lenses, each of which has a certain focal play. Unless the distance between those lenses remains constant, the overall focal length will change. By focusing the lenses independently we create a zoom lens.

As mentioned, the best way to handle this is to bifurcate every time a lens with focal play is introduced. We then create 2 matrices M_1 and M_2.

6.2. **Zoom Range.**

A zoom range can be handled in a similar manner to focal play. In this case, we must construct 4 matrices representing the focal play at each end of the zoom range. Note that many

[23] And many modern camera lenses have 10 or more components.

[24] Generally, only macro lenses have finite maximum object distances.

zoom lenses change size with focal length. As mentioned earlier, this need not concern us as long as we are consistent in our definition of object distances.

The reported data for a lens may only include d_{so} and m_u at a given focal length. Many such lenses are parfocal, and it is a fair assumption that d_{so} is the same throughout the zoom range unless otherwise stated. Although less justified, it also is reasonable to treat the maximum magnification at one end of the range as proportional to that at the other end. For example, for a 70-200mm lens we may be given m_u only at the 200mm end. We then would assume that at the 70mm end $m'_u = \frac{70}{200} \cdot m_u$. Of course, these suppositions could be tested through measurement.

6.3. **Managing Ranges.** In the process of adding components, we may bifurcate at various points into 2 matrices for focal play or 4 for a zoom lens. So what is the final matrix? We need to accommodate all combinations. If we have 2 prime focus lenses and a zoom lens, we would have $2 \times 2 \times 4 = 16$ final matrices, representing 8 focal ranges. In theory we could achieve any value in these by careful manual adjustment of zoom and focus knobs. In practice, we really only care about the extreme values. That is, what can we maximally achieve with a given configuration? It almost always suffices to determine the maximum magnification, minimum object distance, maximum focal range, and so on. A simple comparison of the values extracted from each of the final matrices yields this.

6.4. **Tracking Autofocus.** One consideration in stacking components is the loss of autofocus. Even if components pass electronic information through, only one lens can autofocus. This alone may not be sufficient to achieve focus depending on the other settings. In any configuration with more than one lens, completely manual focusing is necessary. Reversed components always fail to pass through electronic information[25]. Moreover, AF mechanisms are not designed to handle the extra weight associated with stacked components. They could be damaged, and the use of AF is discouraged even when available[26]. To summarize, *AF and IS should be turned off on all lenses in a stack.*

6.5. **Physical Considerations.** There are a number of physical considerations as well.

- Just because a configuration is possible in theory doesn't mean it can be achieved in practice. Lenses may physically fail to connect or lack sufficient space for reversal. In particular, ultra wide-angle lenses sometimes have a protruding front piece that makes stacking infeasible.
- Almost all adapter ring types are available except for those with a female bayonet on one end. Nikon has a version, Canon does not.
- Some cameras (Canon EOS cameras in particular) have firmware that is error prone when one of their telextenders is used in a non-standard configuration. Unless it detects a proper AF lens, the software will prevent a photo from being taken.
- Top mounted flashes may be obstructed by a lengthy stack of components. While a ring flash may help, the cord connecting it to the camera may not be long enough in some cases.
- Torque is an issue. Aside from the difficulty of maneuvering and holding a large configuration, we must be careful of the strain placed on the connection to the camera (as well as any rings or other connectors along the way). It is best to support the stack of components, either through the use of bracing rings (such as for a telescope) or by holding it with one hand. A camera with a long stack should never be placed on a tripod without some additional support. Ideally, the mount point would be somewhere along the stack as with a large telephoto lens.

[25] Actually, at least one vendor produces a special mount connected by a wire that allows a reversed lens to autofocus. However, it is quite expensive.

[26] A similar consideration applies to image stabilization. IS should be deactivated when stacking components.

6.6. **Estimating F-stop.** As mentioned, reversing a lens reduces the aperture – though by how much is hard to tell. A reasonable approach is to use the ratio of the diameters of the front and rear pieces of glass (if the front piece is larger than the rear). The aperture decreases by this factor, and the F-stop increases by it. For example, consider an F/1.8 lens with a rear piece of glass 10mm in diameter and a front piece 30mm in diameter. We would reduce the aperture by a factor of 3, corresponding to a tripling of the F-stop to F/5.4.

While it is a fair bet that intermediate components such as adapter rings or extension tubes will not affect the already reduced light cone, it is possible that a front-mount piece could. If a component reduces the probable width of the light cone, then we should reduce the aperture by a corresponding factor.

Returning to our previous example, let us suppose that the front end of our system consists of a 62mm wide closeup lens (such as the Nikon 5T) on a Canon 70-200 f/4L zoom lens. The aperture (diameter) drops to 0.92537 of its original value. This means a reduction in effective area of 0.8563. The F-stop correspondingly increases to 4.3226. Note that for a 1.6X crop sensor, the effective F-stop (that corresponding to light that hits the sensor) is then 6.916.

6.7. **Aberrations.** In addition to a reduction in quality with each glass component, aberrations can accumulate as well. First order optics doesn't consider these. A typical lens is carefully engineered to minimize aberrations in one direction. Reversing it may have the opposite effect (or not, depending on the aberration). Combining lenses may compound or reduce the aberrations. There are some rules of thumb, but the only way to tell is through trial and error. On the upside, some interesting and unexpected effects can result.

It also is important to note that many lenses were designed for full-frame cameras. Sometimes this helps and sometimes it hurts when switching them around.

Last, if the light beam encounters obstructions, the resulting image may be confined to a circular region or some other subset of the ordinary image rectangle.

7. An Example

Let us consider an example of how to compute the characteristics of a complex system. Suppose that our system consists of the following components in order from the camera forward:

(1) Canon 1.4X telextender.
(2) Canon 70-200mm F/4L zoom lens with a reported closest object distance of 1.2m and a maximum magnification of 0.21 at 200mm zoom. The unit is 172mm long, invariant while focusing or zooming.
(3) Reversing and adapter rings that are 0.5 cm total thickness.
(4) Canon 28mm F/2.8 prime focus lens with a reported 0.3m closest object distance and a maximum magnification of 0.13. The unit is approximately 62.5mm long at its maximum, but varies in length while focusing.

We assume that $\delta = 44mm$, as for a Canon EF-S camera.

7.1. **Stage 1: 28mm Lens.** At the near focus, we use $f = 28$, $\tilde{d}_{so} = 300$, $L = 62.5$, and $m_u = 0.13$ (all as reported in the manual) in equation 3 to get:

$$M_1 = \begin{pmatrix} 1.4414 & 59.5451 \\ -0.0357 & -0.7816 \end{pmatrix}$$

At the far end, we use equation 5 to get:

$$M_2 = \begin{pmatrix} 1.5714 & 68.1101 \\ -0.0357 & -0.9116 \end{pmatrix}$$

As expected, these do not differ much.

Because the lens has been reversed, we use instead:

$$M_1^r = \begin{pmatrix} -0.7816 & 59.5451 \\ -0.0357 & 1.4414 \end{pmatrix}$$
$$M_2^r = \begin{pmatrix} -0.9116 & 68.1101 \\ -0.0357 & 1.5714 \end{pmatrix}$$

At the end of this stage, our set of matrices is:

$$M = \{M_1^r, M_2^r\}$$

7.2. **Stage 2: Adapter Rings.** We need a space of $5mm$ so we use

$$M_3 = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$$

At the end of this stage, our set of matrices is:

$$M = \{M_3 M_1^r, M_3 M_2^r\}$$

7.3. **Stage 3: 70-200mm Lens.** For a zoom lens, we need 4 matrices – two at each end of the focal range.

On the 200mm end, we use $f = 200$, $\tilde{d}_{so} = 1200$, $L = 172$, and $m_u = 0.21$ to get a near focus matrix

$$M_4 = \begin{pmatrix} 0.0100 & 199.6838 \\ -0.0050 & 0.1581 \end{pmatrix}$$

and a far focus matrix

$$M_5 = \begin{pmatrix} 0.2200 & 202.2838 \\ -0.0050 & -0.0519 \end{pmatrix}$$

On the 70mm end we use $f = 70$, $\tilde{d}_{so} = 1200$, $L = 172$, and the scaling assumption of section 6.2 to get $m_u = 0.0735$. The near matrix is:

$$M_6 = \begin{pmatrix} 0.5551 & 52.4492 \\ -0.0143 & 0.4517 \end{pmatrix}$$

and the far one is:

$$M_7 = \begin{pmatrix} 0.6286 & 53.3592 \\ -0.0143 & 0.3782 \end{pmatrix}$$

At the end of this stage, our set of matrices is

$$M = \{M_4 M_3 M_1^r, M_4 M_3 M_2^r, M_5 M_3 M_1^r, M_5 M_3 M_2^r, M_6 M_3 M_1^r, M_6 M_3 M_2^r, M_7 M_3 M_1^r, M_7 M_3 M_2^r\}$$

7.4. **Stage 4: Telextender.** The matrix for the $1.4X$ telextender is

$$M_8 = \begin{pmatrix} 1.4 & 30.1714 \\ 0 & 0.7143 \end{pmatrix}$$

as discussed in section 5.1.

The final set of matrices then is

$$\begin{aligned} M = \{&M_8 M_4 M_3 M_1^r, M_8 M_4 M_3 M_2^r, \\ &M_8 M_5 M_3 M_1^r, M_8 M_5 M_3 M_2^r, \\ &M_8 M_6 M_3 M_1^r, M_8 M_6 M_3 M_2^r, \\ &M_8 M_7 M_3 M_1^r, M_8 M_7 M_3 M_2^r\} \end{aligned}$$

7.5. **System Parameters.** In Table 1 we extract the parameters of the effective lens associated with each of the 8 final matrices using the method in section 3.3.

TABLE 1. System Parameters Associated with the 8 End Matrices

Matrix	28mm Object	Zoom	Zoom Object	f	$\tilde{d}_{fo}$	m_u	Measured $\approx m_u$
$M_8 M_6 M_3 M_1^r$	Near	70	Near	579.5838	39.5409	3.5174	3.71
$M_8 M_7 M_3 M_1^r$	Near	70	Far	-6683.4278	40.3600	3.5000	3.55
$M_8 M_4 M_3 M_1^r$	Near	200	Near	1655.9536	39.5409	10.0497	10.09
$M_8 M_5 M_3 M_1^r$	Near	200	Far	-210.3819	40.3600	10.0000	9.65
$M_8 M_6 M_3 M_2^r$	Far	70	Near	2507.2361	43.1777	3.5040	3.87
$M_8 M_7 M_3 M_2^r$	Far	70	Far	-677.4359	44.0000	3.5000	3.55
$M_8 M_4 M_3 M_2^r$	Far	200	Near	7163.5317	43.1777	10.0115	10.09
$M_8 M_5 M_3 M_2^r$	Far	200	Far	-191.6610	44.0000	10.0000	9.65

A maximum magnification of around 10X is obtained when we use the 200mm end of the zoom lens with any combination of focusing. By comparison, the rather expensive Canon MP-E65 macro lens provides 5x magnification (albeit at a much better F/2.8).

As a side note, the front piece of glass on the 28mm lens is approximately 2X the diameter of the rear piece. So, using our rule of thumb we take the F/2.8 and convert it to F/5.6. This is higher than the F/4 of the 70-200mm lens (which is forward in orientation), so we use it. None of the external physical elements (such as adapter rings) are expected to reduce aperture in this configuration. The telextender reduces the aperture by a multiple of 1.4, yielding F/7.84. If we use this with a 1.6X cropped sensor, the best case F-stop is then F/12.544. We likely will encounter much worse.

While the 200mm zoom produces a uniform image, the 70mm end of the zoom produces a circular image area. This implies that some sort of obstruction in the light path comes into play in this configuration.

The empirical magnifications in table 1 were obtained using a cropped frame sensor, while our computed m's speak to a full-frame sensor. This is of no consequence. Though the process of measuring m is dependent on the sensor size, m itself is not.

A few aspects of our results appear suspicious and bear examination:

- The focal lengths look absurd. This is an unfortunate peculiarity of our particular configuration. The formula for thin lenses is $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}$ with d the distance between them. A similar formula holds for thick lenses with d the distance between the front principal plane of the rear lens and the rear principal plane of the front lens. We haven't included them here, but the cardinal points happen to fall out so that the forward focal point of the zoom lens is around 28mm in front of the canister, while the rear focal point of the reversed 28mm lens is around 25mm inside of its canister. The two focal points almost overlap and $d \approx f_1 + f_2$. This is the condition under which the combined focal length diverges. This isn't a problem, and other parameters are well behaved (in fact, m_u and d_o are remarkably stable). However, it does mean that the total f is extraordinarily sensitive to initial conditions. A slight change in the input m_u values or the size of the spacer ring will cause dramatic changes in f. As a result these particular values of f should not be taken too seriously.
- When the zoom lens is at far focus, the computed magnifications are exactly 10 and 3.5. Also, $\tilde{d}_{fo} = 44$ when both lenses have a far focus. These round numbers are suspicious. However, we are not in error. In some sense, the configuration behaves like a microscope, with an objective and an eyepiece. In order to achieve focus on the sensor, light entering the front of the zoom lens must be from an infinitely distant object – that is, parallel. So, the light emerging from the reversed 28mm lens must be parallel. However, the 28mm lens in a forward configuration would focus light from far away to a plane 44mm behind the unit. So light from an object 44mm in front of the reversed lens would emerge parallel from the other side. The magnification is the ratio of parallel beam widths. It is not hard to see that this is the ratio of focal lengths. Accounting for the telextender, at the 70mm end the ratio is $\frac{(1.4)\cdot 70}{28} = 3.5$ and at the 200mm end it is $\frac{(1.4)\cdot 200}{28} = 10$. This also explains the $\tilde{d}_{fo} = 44$ number, precisely where the image sensor would be on the non-reversed 28mm lens.
- The computed object distance $\tilde{d}_{fo}$ is independent of the zoom for any given choice of near or far focusing of the two lenses. This is not a result of our assumption that m_u scales with f for the zoom lens; it holds even if we vary the value. Rather, it probably follows because the input $\tilde{d}_{fo}$ for the 70-200mm lens is the same at both ends of the zoom range.

8. Summary Tables

First, our notation (all distances are in mm).

- Matrix: $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$.
- δ: Distance from camera sensor to rear of lens. Fixed for a camera family.
- w_s: Sensor dimension in the relevant direction (vertical, horizontal, or diagonal).
- L: Length of the lens or stack measured from the seal with the camera to the frontmost point.
- f: Focal length.
- $\tilde{d}_{fo}$: Minimum distance from front of forward-most component to the object.
- $\tilde{d}_{so}$: Minimum distance from sensor to the object. Assumed to coincide with $\tilde{d}_{fo}$, so $\tilde{d}_{so} = \tilde{d}_{fo} + L + \delta$.
- m_u: Upright maximum magnification (assumed at $\tilde{d}_{fo}$).
- m_o: Optical maximum magnification (assumed at $\tilde{d}_{fo}$). Related by $m_o = -m_u$.
- FOV: Field of view (in same direction as w_s).

TABLE 2. Component → Matrix

Item	Given	A	B	C	D
Lens (d_o finite)	$f, m_u, L, \tilde{d}_{fo}$	$\frac{\delta}{f} - m_u$	$(\tilde{d}_{so} - L - \delta) \cdot m_u + \frac{\delta}{m_u} - \frac{\delta \cdot (\tilde{d}_{so} - L - \delta)}{f}$	$-\frac{1}{f}$	$\frac{(\tilde{d}_{so} - L - \delta)}{f} - \frac{1}{m_u}$
Lens (d_o infinite)	f, L, m_u at any finite $\tilde{d}_{so}$	$\frac{\delta}{f}$	$f + m_u\delta + \frac{\delta}{m_u} - \frac{\delta(\tilde{d}_{so} - L - \delta)}{f}$	$-\frac{1}{f}$	$\frac{(\tilde{d}_{so} - L - \delta)}{f} - m_u - \frac{1}{m_u}$
Lens (Deduced Low End of Zoom Range)	f, other end's $\tilde{d}_{so}$ and m_u, ratio $r = \frac{f_{this}}{f_{other}}$	$\frac{\delta}{f} - r \cdot m_u$	$r \cdot (\tilde{d}_{so} - L - \delta) \cdot m_u + \frac{\delta}{r \cdot m_u} - \frac{\delta \cdot (\tilde{d}_{so} - L - \delta)}{f}$	$-\frac{1}{f}$	$\frac{(\tilde{d}_{so} - L - \delta)}{f} - \frac{1}{r \cdot m_u}$
Telextender	Multiplier x	x	$\delta(x - \frac{1}{x})$	0	$\frac{1}{x}$
Extension Tube	Thickness t	1	t	0	1
Ring (non-trivial thickness)	Thickness t	1	t	0	1
Closeup Lens	Diopter D	1	0	$\frac{-D}{1000}$	1
Reversed Lens	ABCD of the Lens	D	B	C	A

TABLE 3. Matrix → Lens

Case	f	d_{fo}	m_u	FOV
$A + \delta C \neq 0$	$-\frac{1}{C}$	$-\left(\frac{B+\delta D}{A+\delta C}\right)$	$-(A + \delta C)$	–
$A + \delta C = 0$	$-\frac{1}{C}$	∞	0	$-w_s \cdot C$

References

Some Useful Texts

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